

The University of Chicago

THE PROBLEM OF BOLZA
FOR DOUBLE INTEGRALS IN THE
CALCULUS OF VARIATIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1939

By

EDWARD ALFRED NORDHAUS

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THE PROBLEM OF BOLZA FOR DOUBLE INTEGRALS
IN THE CALCULUS OF VARIATIONS

1. Introduction. The problem of the calculus of variations discussed in this paper is an extension to double integrals of the problem of Bolza for simple integrals [4]¹. Closely related questions have been considered by Hilbert [9] and more recently by Tucker [14], who has used the methods of tensor analysis. Stated briefly, the problem is that of finding in a class of admissible surfaces $z = z(x, y)$, whose boundary curves are required to lie on a fixed capstan-shaped surface, one which minimizes a sum of the form

$$I = \iint_A f(x, y, z, p, q) dx dy + \int_L g(x, y, z, x', y', z') ds.$$

The double integral is taken over a region A which is the projection of a portion of an admissible surface on the xy -plane and the line integral is evaluated in the positive sense along the boundary curve L of that surface. This problem is also a generalization of a problem discussed by Bolza [5] and by Simmons [12] of minimizing a double integral with variable limits. In Section 2, definitions and hypotheses sufficient for an analysis of the problem are given. These permit a more precise statement of the problem than that given above. A coordinate system which is of service throughout

¹ The numbers in square brackets refer to the bibliography at the end of the paper.

the paper is defined in Section 3. By means of a slight modification of the method used by Simmons for differentiating a double integral with variable limits, a differentiation formula for integrals is obtained. This formula, stated in Section 4, is useful in deriving the first and second variations. Section 5 is devoted to the construction of a family of admissible surfaces embedding a given surface and having prescribed variations along the boundary curve. A converse problem is also treated. The first variation and an associated transversality condition are presented in Section 6. The transversality condition is obtained without the assumption of non-tangency made by Simmons. Of special interest is the calculation of a simple form for the second variation, as shown in Section 7. In Section 8 a Jacobi condition phrased in terms of a boundary value problem is stated. Analogues of the necessary conditions of Weierstrass and Legendre which must hold along the boundary curve of a minimizing surface are given in Section 9. In the final section an application of the results of this paper is made to an illustrative example.

2. Analytic basis. In this section definitions and hypotheses sufficient for an analysis of the problem proposed in the introduction are given, and the problem is then restated.

An arc in xyz-space which is continuous and which consists of a finite number of sub-arcs each of which has a continuously turning tangent, will be termed a regular arc. This implies that on each sub-arc the parameter may be so chosen that the functions $x(u)$, $y(u)$, $z(u)$ defining the arc have continuous derivatives such that $x'^2 + y'^2 + z'^2$ is different from zero. An analogous definition holds for a regular arc in the plane.

A regular surface $z = z(x, y)$, with (x, y) on a region A of the xy -plane, is one for which the arc bounding the region A is regular, and for which the function $z(x, y)$ is single valued and is piecewise of class C^1 , that is, it is continuous and has continuous first partial derivatives in a neighborhood of the region A ; or it is a continuous surface consisting of a finite number of partial surfaces each of which has these properties.¹

It is assumed that there is a five-dimensional open region R_1 of points (x, y, z, p, q) such that the function f is of class C^m in R_1 .

It is further assumed that there is a six-dimensional open region R_2 of points (x, y, z, x', y', z') in which the following conditions are satisfied: (a) if the set (x', y', z') is in R_2 and k is positive, then the set (kx', ky', kz') is in R_2 ; (b) the function g is of class C^m and is positively homogeneous of order one in (x', y', z') for (x, y, z, x', y', z') in R_2 . In our subsequent discussion, a set (x, y, z, x', y', z') will frequently be denoted more simply by (x, x') .

By an admissible surface we shall mean a regular surface defined by an equation of the form

$$(2:1) \quad z = z(x, y) \quad (x, y) \text{ on } A$$

whose elements (x, y, z, z_x, z_y) are interior to R_1 , and whose boundary curve L is a simply closed regular arc with elements (x, x') interior to R_2 . It follows that no two points of L can lie on the same parallel to the z -axis. The orthogonal projection of the curve L

¹A function is said to be of class $C^{(n)}$ in a region R if it is continuous in R and all of its derivatives of orders 1, 2, ..., n are continuous in R .

on the xy -plane is assumed to be simply closed regular arc C bounding the region A . The curve L is said to be described in the positive sense when the curve C is described in the positive sense.

A fixed capstan-shaped surface may be represented in parametric form by the equations

$$(2:2) \quad x = X(u,v), \quad y = Y(u,v), \quad z = Z(u,v),$$

where $0 \leq u \leq \lambda$, $-\epsilon \leq v \leq +\epsilon$. The functions $X(u,v)$, $Y(u,v)$, $Z(u,v)$ are assumed to be of class C^{∞} in their arguments, and to have a positive real period λ in the variable u for values v on the interval $-\epsilon \leq v \leq +\epsilon$. It is also assumed that the values (x,y,z) defined by the equations (2:2) are interior to the region R_2 and that the fixed surface has no singular points for values of u and v under consideration.

The problem of Bolza for double integrals considered in this paper is now that of finding the class of admissible surfaces with equations of the type (2:1) whose boundary curves are required to lie on the fixed surface (2:2), one which gives the functional

$$I = \iint_A f(x,y,z,p,q) dx dy + \int_L g(x,x') ds$$

a minimum value. Here $p = \partial z / \partial x$, $q = \partial z / \partial y$ and the line integral is taken in the positive sense along the boundary curve L of an admissible surface S whose projection on the xy -plane is the region A . The arguments of the function g are those belonging to the curve L , and accents denote derivatives taken with respect to the parameter s , which represents arc length on the boundary curve. As a consequence of the homogeneity condition imposed on

the function g , the value of the line integral is independent of the parametric representation of the boundary curve, provided the positive sense on the curve L is preserved. If the region R_2 of points (x, x') described above is such that the set $(x, -x')$ is in R_2 whenever (x, x') is in R_2 and if the relation $g(x, -x') = g(x, x')$ holds for every set (x, x') in R_2 , then the value of the line integral appearing in the sum I is independent of the orientation of the boundary curve L of the surface S . This is the so-called reversible case.

Throughout this paper we shall be concerned with a surface S_0 of class C'' which we assume minimizes the functional I . Our problem is then that of finding the special properties which a minimizing surface of this type must possess. It will be convenient to assume that the boundary curve L_0 of S_0 is given by equations (2:2) with $v = 0$. Such a representation of the capstan surface may always be obtained provided the normal lines to this surface along the curve L_0 are nowhere parallel to the xy -plane. A simply closed curve L may be defined on the capstan surface by specifying a function $v = v(u)$ which is of class C'' and has a period λ in the variable u . The functional to be minimized then has the form

$$(2:3) \quad I = \iint_A f(x, y, z, p, q) dx dy + \int_0^\lambda g(x, x') du,$$

where the arguments of g are those belonging to the curve L , accents denote derivatives taken with respect to the variable u , and the region A is defined as before.

3. A coordinate system. Let C_0 be a simply closed curve in the xy -plane defined by equations of the form

$$x = x_0(u), \quad y = y_0(u) \quad (0 \leq u \leq \lambda),$$

where the functions $x_0(u)$, $y_0(u)$ are of class C^m , have a period λ in the variable u , and have $x_0'^2 + y_0'^2 \neq 0$ along C_0 . If the parameter u represents arc length on C_0 , then λ is the length of C_0 , and $x_0'^2 + y_0'^2 = 1$. We assume for definiteness that the parameter u measures in the positive direction on C_0 . In a neighborhood of C_0 a uv -coordinate transformation may be defined by the equations

$$(3:1) \quad x = X(u,v), \quad y = Y(u,v) \quad (-\epsilon \leq v \leq +\epsilon).$$

The functions $X(u,v)$, $Y(u,v)$ have the following properties: (1) they are of class C^m in their arguments, (2) they have a period λ in the variable u , (3) the equations $x_0(u) = X(u,0)$, $y_0(u) = Y(u,0)$ are identities in u , (4) the functional determinant

$$(3:2) \quad D(u,v) = \begin{vmatrix} X_u & Y_u \\ X_v & Y_v \end{vmatrix}$$

is different from zero along C_0 ; that is, for $v = 0$. For example, we might take the curves $v = \text{constant}$ as the parallel curves to C_0 and the curves $u = \text{constant}$ as the normals to C_0 . In Simmons' paper [12, p. 237], the positive direction of the variable v was taken along the outer normal to C_0 , and hence the uv -directions were oriented oppositely to that of the xy -directions. The functional determinant (3:2) had the value $D = -(1 + v/\rho)$, where ρ is the radius of curvature of C_0 , and on C_0 , $D = -1$. In this paper, however, the positive direction of the variable v is taken to be directed interior to the region bounded by C_0 , resulting in a similarly directed orientation of the xy -directions and the uv -directions. The determinant D in equation (3:2) is therefore

positive for values of v sufficiently near zero.

Let $v = v(u, a)$ be a function of class C^n for all sets (u, a) with $0 \leq u \leq \lambda$ and the parameter a sufficiently near zero, with the further properties that $v(u, a)$ has the period λ in the variable u , and $v(u, 0) \equiv 0$. It follows that the equations

$$(3:3) \quad x = X[u, v(u, a)], \quad y = Y[u, v(u, a)]$$

define a one parameter family of simply closed curves C_a in the xy -plane for the parameter a near zero. The family contains the curve C_0 for the parameter value $a = 0$, and each curve of the family has the period λ in the variable u . Denote by A_a the area bounded by the curve C_a . The determinant $D[u, v(u, a)]$ obtained by setting $v = v(u, a)$ in equation (3:2) remains positive when the parameter a is sufficiently near zero.

4. A differentiation formula. Before deriving the first and second variations for the functional I defined in Section 2, it is convenient to develop a method for differentiating a sum of the type (2:3) which involves a parameter in the limits of integration of the double integral and in both integrand functions. The procedure employed is a simple modification of a method given by Simmons [12, p. 238].

Let C^* be a simply closed curve defined by the equation (3:1) for a constant positive value $v = v^*$ near zero, and let A^* be the region bounded by C^* . Denote by ΔA the area included between the curves C^* and C_a . Consider the expression

$$(4:1) \quad J(a) = \iint_{A_a} F(x, y, a) dx dy + \int_0^\lambda G(u, a) du,$$

where $F(x, y, a)$ is of class C^n for all sets (x, y, a) with (x, y) in a neighborhood of the area A_0 bounded by C_0 and having the parameter

a sufficiently near zero; and where $G(u, a)$ is of class C^* for all sets (u, a) with $0 \leq u \leq \lambda$ and the parameter a near zero. The single integral is to be evaluated along the curve C_a . We may then write

$$J(a) = \iint_{A^*} F \, dx \, dy + \iint_{\Delta A} F \, dx \, dy + \int_0^\lambda G \, du,$$

where in the first double integral the parameter a occurs only in the integrand. It is only the second double integral whose differentiation presents difficulties. This double integral may be transformed to uv -coordinates by means of equations (3:1) and has the value:

$$(4:2) \quad \int_0^\lambda \int_{v(u,a)}^{v^*} F[X(u,v), Y(u,v), a] D(u,v) \, dv \, du.$$

The derivative of the expression (4:2) is found by the usual formula for differentiating a definite integral with respect to a parameter occurring in the integrand and in the limits of integration to be

$$(4:3) \quad \int_0^\lambda \int_{v(u,a)}^{v^*} \frac{\partial F}{\partial a} D \, dv \, du - \int_0^\lambda F D v_a \, du,$$

where in the single integral the variable v , wherever it occurs, is the function $v(u, a)$. If the double integral appearing in the expression (4:3) is expressed in terms of xy -coordinates, a formula for $J'(a)$ is obtained.

LEMMA 4:1. The first derivative of the expression $J(a)$ defined in equation (4:1) has the value

$$(4:4) \quad J'(a) = \iint_{A_a} \frac{\partial F}{\partial a} \, dx \, dy + \int_0^\lambda \left(\frac{\partial G}{\partial a} - F D v_a \right) du.$$

An explicit expression for $J''(a)$ may similarly be calculated, but will not be used in deriving the second variation for the

functional I defined by equation (2:3), since it will be shown that it is much simpler to calculate the second variation of I after the transversality condition associated with the first variation has been established.

5. Embedding theorems. When a function $v(u,a)$ having the properties described in Section 3 is substituted for v in the equation (2:2) of the fixed surface, a one parameter family of simply closed curves L_a is obtained with the equations

$$(5:1) \quad x = X[u, v(u, a)], \quad y = Y[u, v(u, a)], \quad z = Z[u, v(u, a)].$$

This family contains a curve L_0 for $a = 0$. The variations of the family along L_0 have the form

$$(5:2) \quad \xi_1 = X_v v_a, \quad \xi_2 = Y_v v_a, \quad \xi_3 = Z_v v_a,$$

where the arguments of X_v , Y_v , Z_v , v_a are $(u, 0)$. The functions $\xi_1(u)$, $\xi_2(u)$, $\xi_3(u)$ have the period λ in the variable u . Let S_0 be a surface of class C^n with an equation of the form $z = z(x, y)$, with (x, y) in A , and having the curve L_0 as its boundary. In this section we first establish the following theorem:

THEOREM 5:1. Let $v(u, a)$ be a function of class C^n for all sets (u, a) with $0 \leq u \leq \lambda$ and a near zero such that $v(0, a) = v(\lambda, a)$ and $v(u, 0) \equiv 0$. Then there exists a one parameter family of admissible surfaces $z = z(x, y, a)$, defined and piecewise of class C^n for the parameter a near zero, which contains the given surface S_0 for $a = 0$, intersects the fixed surface (2:2) in the curves of the family (5:1) for corresponding values of the parameter, and has the prescribed variations (5:2) along the boundary curve L_0 on which $a = 0$.

In order to prove this theorem, let v^* be a fixed positive value of v , sufficiently near zero, so the uv -coordinate system is well defined. A fixed simply closed curve L^* lying on the given surface S_0 may be determined by the equations

$$x = X(u, v^*), \quad y = Y(u, v^*), \quad z = z[X(u, v^*), Y(u, v^*)].$$

For a fixed value of the parameter a , we consider a surface having curvilinear coordinates u and w and defined by the equations

$$\begin{aligned} (5:3) \quad x &= (1 - w)X(u, v) + wX(u, v^*), \\ y &= (1 - w)Y(u, v) + wY(u, v^*), \\ z &= (1 - w)Z(u, v) + wz[X(u, v^*), Y(u, v^*)], \end{aligned}$$

where $v = v(u, a)$ and $0 \leq w \leq 1$. The boundary of this surface is composed of the curves L_a and L^* , as is seen by setting $w = 0$ and $w = 1$, respectively. When a value $u = u_0$ on the interval $(0, \chi)$ is selected, the equations (5:3) reduce to the equations of a straight line connecting the points on the curves L_a and L^* for which $u = u_0$.

The surface (5:3) together with the portion of S_0 bounded by the curve L^* defines a surface S_a whose boundary is L_a . It remains to show that the surface S_a is expressible in the form $z = z(x, y, a)$ and is admissible. To do this, it is sufficient to show that the first two equations in (5:3) may be solved for the variables u and w as functions of x, y, a . The functional determinant of these two equations formed with respect to the variables u and w has the value

$$\Delta(u, w, v^*; a) = \begin{vmatrix} (1 - w)(X_u + X_v v_u) + wX_u^*, & X^* - X \\ (1 - w)(Y_u + Y_v v_u) + wY_u^*, & Y^* - Y \end{vmatrix},$$

where the arguments of the unstarred terms in X , Y and their derivatives are u , $v(u, a)$ and of the starred terms are u , v^* .

We may regard v^* as a parameter and obtain by the theorem of mean value the expressions

$$\begin{aligned} X(u, v^*) - X(u, v) &= (v^* - v)X_v[u, v + \theta(v^* - v)], \\ Y(u, v^*) - Y(u, v) &= (v^* - v)Y_v[u, v + \theta(v^* - v)], \end{aligned}$$

where $v = v(u, a)$ and $0 \leq \theta \leq 1$. If these expressions are substituted into the second column of the determinant Δ , we obtain a relation of the form

$$(5:4) \quad \Delta(u, w, v^*, a) = (v^* - v)D(u, w, v^*, a)$$

where

$$D(u, w, v^*, a) = \begin{vmatrix} (1 - w)(X_u + X_v v_u) + wX_u^*, & X_v^* \\ (1 - w)(Y_u + Y_v v_u) + wY_u^*, & Y_v^* \end{vmatrix},$$

and where X_v^* and Y_v^* have the arguments u , $v + \theta(v^* - v)$. When $v(u, 0) \equiv 0$, we have $v_u(u, 0) \equiv 0$ and the determinant $D(u, w, 0, 0)$ is identical with the determinant (3:2) with $v = 0$ and is therefore different from zero. It follows that there is a constant v^* near zero and a constant $d > 0$ such that the expressions $D(u, w, v^*, a)$ and $v^* - v(u, a)$ are different from zero for $0 \leq u \leq \chi$, $0 \leq w \leq 1$, $-d \leq a \leq d$. From the formula (5:4) it is seen that $\Delta(u, w, v^*, a)$ is different from zero for these values of v^* and a . By the use of implicit function theorems the first two equations of (5:3) have solutions

$$u = U(x, y, a), \quad w = W(x, y, a),$$

which are of class C^n in (x,y,a) for the parameter a sufficiently near zero and for values (x,y) in a neighborhood of those belonging to L_0 . If we substitute these functions into the third equation of (5:3), then for a fixed value of a an admissible surface $z = \Psi(x,y,a)$ of class C^n is determined which intersects the fixed surface in the curve L_a . The surface S_a is therefore representable in the form $z = z(x,y,a)$ and is piecewise of class C^n and admissible. As the parameter a varies, a one parameter family of surfaces of the above type is obtained which contains the given surface S_0 for $a = 0$ and has the desired variations (5:2) along L_0 . This completes the proof of Theorem 5:1.

We next prove the following theorem, which is a converse of the preceding:

THEOREM 5:2. Let $z = z(x,y,a)$ be a one parameter family of admissible surfaces piecewise of class C^n for the parameter a near zero, and containing a surface S_0 for $a = 0$ which is nowhere tangent to the fixed surface (2:2) along their intersection. Then the intersection of this family with the fixed surface determines a one parameter family of simply closed curves L_a which is defined by a function $v(u,a)$ of class C^n which has the period λ in the variable u and has $v(u,0) \equiv 0$.

In particular, the family $z = z(x,y,a)$ may be chosen in the form $z = z(x,y) + a \zeta(x,y)$, where $z = z(x,y)$ is the equation for S_0 and $\zeta(x,y)$ is a function piecewise of class C^n .

To prove the theorem, we note that the equation

$$(5:5) \quad z[X(u,v), Y(u,v), a] - Z(u,v) = 0$$

holds for values $0 \leq u \leq \lambda$ when $v = 0$, $a = 0$, since the boundary curve L_0 of S_0 is given by equations (2:2) with $v = 0$. The derivatives of the left member of this equation with respect to the

variables u and v are the expressions

$$(5:6) \quad \rho(u, v, a) = pX_u + qY_u - Z_u,$$

$$(5:7) \quad \sigma(u, v, a) = pX_v + qY_v - Z_v.$$

Since the surface S_0 is by hypothesis nowhere tangent to the surface (2:2) along the curve L_0 , the quantities $\rho(u, 0, 0)$, $\sigma(u, 0, 0)$ cannot vanish simultaneously at any point on L_0 . Since $\rho(u, 0, 0) \equiv 0$, it follows that $\sigma(u, 0, 0) \neq 0$. By the extended implicit function theorem [2], the equation (5:5) has a solution $v = v(u, a)$ of class C^n on $0 \leq u \leq \lambda$, $|a| < a_0$. As a result of the initial conditions, one has $v(u, 0) \equiv 0$. Furthermore, since the functions $X(u, v)$, $Y(u, v)$, $Z(u, v)$ have a period λ in the variable u , the function $v(u, a)$ also has this periodicity property. It is to be noted that the expression $\sigma(u, v(u, a), a)$ is different from zero when the parameter a is sufficiently small.

6. The first variation. In order to obtain the first variation of the functional I to be minimized, we suppose that a one parameter family of admissible surfaces S_a of the type

$$(6:1) \quad z = z(x, y, a)$$

has been constructed containing a surface S_0 for $a = 0$, and is defined for the parameter a near zero. This family intersects the fixed surface (2:2) in a family of simply closed curves L_a with the equations

$$(6:2) \quad x = X[u, v(u, a)], \quad y = Y[u, v(u, a)], \quad z = Z[u, v(u, a)],$$

where the function $v(u, a)$ has the properties described in Section 3. The possibility of the construction of the families (6:1) and (6:2) has been discussed in Theorems 5:1 and 5:2 of the preceding section.

We further assume that $z_a(x, y, a)$ is piecewise of class C'' for the parameter a near zero.

Before deriving the expression for the first variation, we develop a number of useful formulas. If we substitute the function $v = v(u, a)$ into equation (5:5) and differentiate the resulting identity with respect to the variables a and u , we obtain the important relations

$$(6:3) \quad z_a = -\sigma v_a, \quad \rho = -\sigma v_u,$$

where the expressions for ρ and σ are given in equations (5:6) and (5:7). At $a = 0$ the relations $v_u = 0$, $\rho = 0$ are satisfied. It is to be noted from the first equation in (6:3) that the variations z_a and v_a are not both arbitrary when $\sigma \neq 0$.

The direction parameters of the normal line to the fixed surface (2:2) at any point are proportional to the determinants

$$D_1 = \begin{vmatrix} Y_u & Z_u \\ Y_v & Z_v \end{vmatrix}, \quad D_2 = \begin{vmatrix} Z_u & X_u \\ Z_v & X_v \end{vmatrix}, \quad D = \begin{vmatrix} X_u & Y_u \\ X_v & Y_v \end{vmatrix},$$

where the arguments of the derivatives of X , Y , Z are (u, v) . Since by hypothesis $D(u, v)$ is different from zero when $v = 0$, we may apply implicit function theorems and solve the first two equations in (2:2) for u and v as functions of x and y . The fixed surface (2:2) may therefore be represented in the non-parametric form $\Phi(x, y, z) = 0$ in a neighborhood of the curve on which $v = 0$. Since $\Phi(X, Y, Z) = 0$ is an identity in u and v we obtain by differentiation the orthogonality relations

$$\begin{aligned} \Phi_x X_u + \Phi_y Y_u + \Phi_z Z_u &= 0, \\ \Phi_x X_v + \Phi_y Y_v + \Phi_z Z_v &= 0. \end{aligned}$$

If these equations are solved for ϕ_x and ϕ_y , we find the relations $D\phi_x = D_1\phi_z$, $D\phi_y = D_2\phi_z$. If ϕ_z were zero, then ϕ_x and ϕ_y are also zero, contrary to the assumption that the fixed surface is free of singular points. If we set $p_1 = -\phi_x/\phi_z$, $q_1 = -\phi_y/\phi_z$ we obtain the relations

$$(6:4) \quad -Dp_1 = D_1, \quad -Dq_1 = D_2,$$

where $(p_1, q_1, -1)$ are now direction parameters for the normal line to the fixed surface (2:2) at any point. It is important to note that these equations are valid when v is replaced by $v(u, a)$, provided the parameter a is sufficiently small. If Z_u and Z_v are eliminated from the determinants D_1 and D_2 in equations (6:4) by means of the relations (5:6) and (5:7) for ρ and σ , we obtain the equations

$$(6:5) \quad \begin{aligned} \sigma X_u - \rho X_v &= Dq, \\ \sigma Y_u - \rho Y_v &= -DP, \end{aligned}$$

where $P = p - p_1$, $Q = q - q_1$. We may now write

$$(6:6) \quad \rho = PX_u + QY_u, \quad \sigma = PX_v + QY_v.$$

Since $\rho = -\sigma v_u$, equations (6:5) may be written in the useful form

$$(6:7) \quad \begin{aligned} \sigma X' &= \sigma (X_u + X_v v_u) = Dq, \\ \sigma Y' &= \sigma (Y_u + Y_v v_u) = -DP, \end{aligned}$$

where accents denote derivatives taken with respect to the variable u . When $a = 0$, equations (6:7) reduce to $\sigma X_u = Dq$, $\sigma Y_u = -DP$.

If the sum I defined in equation (2:3) is evaluated for a surface of the family (6:1), an expression of the form studied in

Section 4 is obtained, since the limits of integration of the single integral in equation (2:3) may be chosen to be independent of the parameter a . On applying the result stated in equation (4:4), we find the expression

$$(6:8) \quad I'(a) = \iint_{A_a} \frac{\partial f}{\partial a} dx dy + \int_0^{\gamma} \left(\frac{\partial g}{\partial a} - f Dv_a \right) du,$$

where A_a is the area bounded by the projection on the xy -plane of the curve of intersection L_a of the fixed surface with that surface of the family (6:1) whose parameter value is a . The arguments of the derivatives of f and g are, respectively, those belonging to the surface S_a and its boundary curve L_a . Throughout the single integral the variable v , wherever it occurs, is the function $v(u, a)$. The Euler expression has the definition

$$(6:9) \quad R(x, y, a) = f_z - \frac{\partial}{\partial x} f_p - \frac{\partial}{\partial y} f_q,$$

where the arguments of the derivatives of f are those belonging to the surface S_a . It follows from the theory of the problem of minimizing a double integral with a fixed boundary curve [3, p.5] that R vanishes at $a = 0$ when S_0 is a minimizing surface. It is of course necessary that this condition hold along a minimizing surface for the problem stated in Section 2. By means of the relation

$$\frac{\partial f}{\partial a} = z_a R + \frac{\partial}{\partial x} (f_p z_a) + \frac{\partial}{\partial y} (f_q z_a)$$

and an application of Green's Theorem, the double integral in equation (6:8) has the value

$$\iint_{A_a} z_a R dx dy + \int_0^{\gamma} z_a (f_p Y' - f_q X') du,$$

provided the surface S_a is piecewise of class C^n for values (x, y) in A_a . We may write

$$\frac{\partial G}{\partial a} = \sum (g_x x_a + g_x, x_a'),$$

where the sum indicated represents a sum of three terms obtained by replacing the letter x by x, y, z in turn. From the relations $x_a = X_v v_a$, $x_a' = X_v v_a' + X_v' v_a$ and similar ones for y and z , it follows that $\partial g / \partial a$ may be written in the form

$$\frac{\partial G}{\partial a} = v_a \sum (g_x - \frac{d}{du} g_x,') X_v + \frac{d}{du} (v_a \sum g_x, X_v).$$

The last term of this expression vanishes when integrated between the limits 0 and λ , since there are no corners on the boundary curve and the functions occurring all have the period λ in the variable u . If we collect the terms of the expression (6:8), use $z_a = -\sigma v_a$ and the relations (6:7), we obtain the following result:

THEOREM 6:1. The first derivative of the expression $I(a)$, when taken over that portion of the surface S_a which is bounded by the fixed surface (2:2), has the value

$$(6:10) \quad I'(a) = \iint_{A_a} z_a R \, dx \, dy + \int_0^\lambda v_a T \, du,$$

where R is defined in equation (6:9) and $T(u, a)$ is the expression

$$(6:11) \quad T = D(Pf_p + Qf_q - f) + \sum (g_x - \frac{d}{du} g_x,') X_v,$$

and $P = p - p_1$, $Q = q - q_1$.

If S_0 is a minimizing surface of class C^n for the functional I , then a first necessary condition is $I'(0) = 0$, and as previously mentioned, $R = 0$. Since by Theorem 5:1 the variation v_a is an arbitrary function of class C^n , it follows from the fundamental lemma for simple integrals that $T = 0$ along the boundary

curve L_0 of S_0 . Hence we have

THEOREM 6:2. At each point on the boundary curve L_0 of a surface S_0 of class C^* which minimizes the functional I , the transversality condition $T = 0$ must be satisfied, where T is defined by equation (6:11) with $a = 0$.

If the fixed capstan surface (2:2) is represented in the non-parametric form $\Phi(x, y, z) = 0$, then by means of the relations $p_1 = -\Phi_x/\Phi_z$, $q_1 = -\Phi_y/\Phi_z$, the transversality condition may be written in the form

$$D[f_p\Phi_x + f_q\Phi_y + (pf_p + qf_q - f)\Phi_z] + \Phi_z \sum (g_x - \frac{d}{du} g_x)X_v = 0.$$

If the function g is identically zero, the non-zero factor D may be deleted and the transversality condition reduces to that given by Simmons [12, p. 245]. An alternative form for the transversality condition in this case is $f = (p - p_1)f_p + (q - q_1)f_q$. It follows that the minimizing surface cannot be tangent to the capstan surface if the function f is different from zero along L_0 . This remark is justifiable since an examination of the derivation of the transversality condition $T = 0$ shows that no use has been made of the non-tangency assumption stated in Theorem 5:2.

As a consequence of the homogeneity condition imposed on the function g , the identity

$$g(x, x') = \sum x' g_{x'}(x, x')$$

holds. If this identity is differentiated with respect to u and the set (x, x') is replaced by the set $[X(u, 0), X_u(u, 0)]$, we obtain the relation $\sum \lambda X_u = \lambda X_u + \mu Y_u + \nu Z_u = 0$, where

$$(6:12) \quad \lambda = g_x - \frac{d}{du} g_{x'}, \quad \mu = g_y - \frac{d}{du} g_{y'}, \quad \nu = g_z - \frac{d}{du} g_{z'},$$

are abbreviations for the Euler directions. If the function f is identically zero, then the transversality condition reduces to

$\sum \lambda X_v = 0$. From the relations $\sum \lambda X_u = 0$, $\sum \lambda X_v = 0$, it is seen that the Euler directions λ, μ, ν coincide with the direction of the normal line to the fixed surface (2:2), unless they vanish simultaneously. At any point on the boundary curve L_0 , the direction parameters of the normal line to the fixed surface, considered in the non-parametric form $\Phi(x, y, z) = 0$, are proportional to the derivatives Φ_x, Φ_y, Φ_z . There then exists a proportionality factor $\tau(u)$ such that the relations

$$(6:13) \quad \lambda = \tau \Phi_x, \quad \mu = \tau \Phi_y, \quad \nu = \tau \Phi_z$$

hold along L_0 . Define $G(x, x') = g(x, x') - \tau \Phi$. It is readily verified that the equations (6:13) can be written in the form

$$(6:14) \quad G_x - \frac{d}{du} G_{x'} = 0, \quad G_y - \frac{d}{du} G_{y'} = 0, \quad G_z - \frac{d}{du} G_{z'} = 0.$$

This proves the following result:

THEOREM 6:3. The Euler equations (6:14) for the problem of minimizing the functional I defined by equation (2:3) when f is identically zero, follow as a consequence of the transversality condition described in Theorem 6:2.

7. The second variation. In this section we make the non-tangency assumption stated in Theorem 5:2. The differentiation formula of Section 4 may be applied to the expression for the first variation given in equation (6:10). We then obtain

$$I''(a) = \iint_{A_a} (z_{aa}R + z_a \frac{\partial R}{\partial a}) dx dy + \int_0^{\chi} (v_{aa}T + v_a \frac{\partial T}{\partial a} - z_a R Dv_a) du.$$

If S_0 is a minimizing surface for the functional I , then for $a = 0$ the relations $R = 0$, $T = 0$, $I''(0) \geq 0$ must be satisfied. The above

equation therefore reduces to

$$(7:1) \quad I''(0) = \iint_{A_0} z_a \frac{\partial R}{\partial a} dx dy + \int_0^{\lambda} v_a \frac{\partial T}{\partial a} du.$$

Let $2\omega(x, y, \zeta, \pi, \kappa)$ be the quadratic form

$$(7:2) \quad 2\omega = f_{zz}\zeta^2 + f_{pp}\pi^2 + f_{qq}\kappa^2 + 2(f_{zp}\zeta\pi + f_{pq}\pi\kappa + f_{qz}\kappa\zeta),$$

where $\zeta(x, y) = z_a(x, y, 0)$, $\pi(x, y) = \zeta_x$, $\kappa(x, y) = \zeta_y$. By differentiation the Euler expression (6:9) with respect to the parameter a it is readily established that for $a = 0$ the Jacobi expression

$$(7:3) \quad J = \omega_\zeta - \frac{\partial}{\partial x} \omega_\pi - \frac{\partial}{\partial y} \omega_\kappa$$

is equal to $\partial R / \partial a$. As a consequence of the homogeneity property

$$2\omega = \zeta \omega_\zeta + \pi \omega_\pi + \kappa \omega_\kappa,$$

the functions J and 2ω are related by the equation

$$(7:4) \quad \zeta J = 2\omega - \frac{\partial}{\partial x} (\zeta \omega_\pi) - \frac{\partial}{\partial y} (\zeta \omega_\kappa).$$

If this value for ζJ is substituted into equation (7:1), then by an application of Green's Theorem and by making use of the first equation in (6:3) and the equation (6:7), the expression for $I''(0)$ may be put in the form

$$(7:5) \quad I''(0) = \iint_{A_0} 2\omega dx dy + \int_0^{\lambda} v_a \left[\frac{\partial T}{\partial a} - D(P\omega_\pi + Q\omega_\kappa) \right] du.$$

The equation (6:11) may be written $T(u, a) = DA + B$, where

$$(7:6) \quad A = Pf_p + Qf_q - f, \quad B = \sum \lambda X_v,$$

and λ, μ, ν are defined in equations (6:12). If we differentiate equation (6:11) with respect to a , we find

$$(7:7) \quad \frac{\partial T}{\partial a} = (D_v A + DA_v) v_a + DA_a + \frac{\partial B}{\partial a}.$$

Here A_a represents the derivative of A with respect to the parameter a wherever it occurs explicitly, and has the value

$$A_a = P \omega_\pi + Q \omega_\kappa - f_z \zeta.$$

This may be shown by differentiating the first equation in (7:6)

with respect to a and using the relations $f_{pa} = \omega_\pi$, $f_{qa} = \omega_\kappa$,

$f_a = f_z \zeta + f_p \pi + f_q \kappa$, $P_a = p_a = \pi$, $Q_a = q_a = \kappa$. The derivatives

ω_π and ω_κ are readily computed from equation (7:2). If we use the relation $v_a = -\zeta/\sigma$, where $\sigma = PX_v + QY_v$, the expression

(7:5) for $I''(0)$ becomes

$$(7:8) \quad I''(0) = \iint_{A_0} 2\omega \, dx \, dy + \int_0^{\lambda} \frac{\zeta^2}{\sigma^2} [D_v A + D(A_v + \sigma f_z)] \, du + \int_0^{\lambda} v_a \frac{\partial B}{\partial a} \, du.$$

We proceed to evaluate the last integral in (7:8). From the second equation in (7:6) we obtain

$$\frac{\partial B}{\partial a} = v_a \sum \lambda X_{vv} + \sum \lambda_a X_v.$$

The expression $\sum \lambda_a X_v$ may be written as

$$\sum \lambda_a X_v = \sum X_v \frac{\partial g_x}{\partial a} + \sum X_v' \frac{\partial g_{x'}}{\partial a} - \frac{d}{du} \sum X_v \frac{\partial g_{x'}}{\partial a}.$$

If we use this relation, then the term $v_a (\partial B / \partial a)$ occurring in the equation (7:8) for the second variation has the value

$$(7:9) \quad v_a \frac{\partial B}{\partial a} = v_a^2 \sum \lambda X_{vv} + \sum X_v v_a \frac{\partial g_x}{\partial a} + \sum (X_v' v_a + X_v v_a') \frac{\partial g_{x'}}{\partial a} - \frac{d}{du} (v_a \sum X_v \frac{\partial g_{x'}}{\partial a}).$$

The last expression in the right member of this equation vanishes when integrated between the limits 0 and λ , since each term has

the period λ in the variable u . The variations $\xi_1 = X_v v_a$,

$\xi_2 = Y_v v_a$, $\xi_3 = Z_v v_a$ have the derivatives

$$\xi_1' = X_v' v_a + X_v v_a', \quad \xi_2' = Y_v' v_a + Y_v v_a', \quad \xi_3' = Z_v' v_a + Z_v v_a'.$$

Let $2\Omega(\xi, \xi')$ represent a quadratic form in the variables (ξ, ξ') $= (\xi_1, \xi_2, \xi_3, \xi_1', \xi_2', \xi_3')$ whose matrix of coefficients is of the sixth order and consists of the second derivatives of the function $g(x, x')$ with respect to the variables x, y, z, x', y', z' .

By using equation (7:9), the relation $v_a = -\zeta/\sigma$ and the definition $2\Omega(\xi, \xi')$, then the equation (7:8) for the second variation can be put into the form

$$\begin{aligned} I''(0) = \iint_{A_0} 2\omega \, dx \, dy + \int_0^{\lambda} \frac{\zeta^2}{\sigma^2} [D_v A + D(A_v + \sigma f_z) + \sum \lambda X_{vv}] \, du \\ (7:10) \qquad \qquad \qquad + \int_0^{\lambda} 2\Omega(\xi, \xi') \, du, \end{aligned}$$

where 2ω is defined by equation (7:2) and 2Ω is the quadratic form described above.

We note that the double integral occurring in equation (7:10) is the same as that for the double integral problem with fixed limits of integration. In the first simple integral the integrand has the factor ζ^2 but contains no other powers or derivatives of the variation ζ . If we use $\sigma = PX_v + QY_v$, then the expression $A_v + \sigma f_z$ occurring in $I''(0)$ has the value

$$A_v + \sigma f_z = (A_x + Pf_z)X_v + (A_y + Qf_z)Y_v,$$

where explicit expressions for A_x and A_y are

$$\begin{aligned} A_x &= P_x f_p + Q_x f_q + P(f_p)_x + Q(f_q)_x - (f)_x, \\ A_y &= P_y f_p + Q_y f_q + P(f_p)_y + Q(f_q)_y - (f)_y, \end{aligned}$$

The notation $(f)_x$, for example, is an abbreviation for the total derivative of f with respect to x , and has the value

$$(f)_x = f_x + f_z p + f_p r + f_q s,$$

where $r = p_x = z_{xx}$, $s = p_y = q_x = z_{xy}$, $t = q_y = z_{yy}$. If r_1 , s_1 , t_1 represent the corresponding second derivatives for the fixed surface (2:2), considered in non-parametric form, then

$$P_x = r - r_1, \quad P_y = s - s_1 = Q_x, \quad Q_y = t - t_1.$$

It may be verified by using the Euler equation $f_z = (f_p)_x + (f_q)_y$ and the relation $Q_x = P_y$, that we may write

$$\begin{aligned} A_x + P f_z &= P(f_p)_x + Q(f_q)_x - (f)_x + (P f_p)_x + (P f_q)_y, \\ (7:11) \quad A_y + Q f_z &= P(f_p)_y + Q(f_q)_y - (f)_y + (Q f_p)_x + (Q f_q)_y. \end{aligned}$$

With the aid of equation (3:2) defining D , we find

$$D_v = X_u Y_{vv} - Y_u X_{vv} + Y_v X_{uv} - X_v Y_{uv}.$$

Another expression for D_v may be obtained if we specialize the coordinate system described in Section 3. We have a normal coordinate system, in the sense that the variable v measures along the inner normal to C_0 , if we impose the restrictions $D X_v = -Y_u$, $D Y_v = X_u$. By means of these relations and the equation (3:2) for D the equations

$$D^2 = X_u^2 + Y_u^2, \quad X_v^2 + Y_v^2 = 1, \quad X_u X_v + Y_u Y_v = 0$$

are readily established. From the equations (6:7) with $v = v_u = 0$ and the above equations we obtain the additional relations

$$(7:12) \quad \sigma X_v = P, \quad \sigma Y_v = Q, \quad \sigma^2 = P^2 + Q^2,$$

which hold when $v = 0$. An expression for D_v may be calculated at once by differentiating $D^2 = X_u^2 + Y_u^2$ with respect to v .

The derivatives X_{uv} , Y_{uv} may be found from the relations $X_v = P/\sigma$, $Y_v = Q/\sigma$. We then find, on using $\rho = PX_u + QY_u = 0$, $X_u = DQ/\sigma$, $Y_u = -DP/\sigma$ and the relations for P_x , P_y , Q_x , Q_y the expression

$$D_v = (D/\sigma^3)[Q^2(r - r_1) - 2PQ(s - s_1) + P^2(t - t_1)].$$

Define $G(x, x') = g(x, x') - \tau(u)\Phi(x, y, z)$, where $\tau(u)$ is an arbitrary function of u of class C^n with the period λ in the variable u and $\Phi(x, y, z) = 0$ is the equation of the capstan surface (2:2) considered in non-parametric form. If we define

$$\lambda^* = G_x - \frac{d}{du} G_{x'}, \quad \mu^* = G_y - \frac{d}{du} G_{y'}, \quad \nu^* = G_z - \frac{d}{du} G_{z'},$$

then the relations $\lambda^* = \lambda - \tau \Phi_x$, $\mu^* = \mu - \tau \Phi_y$, $\nu^* = \nu - \tau \Phi_z$ are readily established. The expression $(\zeta^2/\sigma^2) \sum \lambda X_{vv}$ occurring in equation (7:10) has the value

$$v_a^2 \sum \lambda X_{vv} = v_a^2 [\sum \lambda^* X_{vv} + \tau \sum \Phi_x X_{vv}].$$

From the identity $\sum \Phi_x X_v = 0$, we obtain by differentiation the relation $\sum \Phi_x X_{vv} = -\sum \Phi_{xv} X_v$, where

$$\Phi_{xv} = \Phi_{xx} X_v + \Phi_{xy} Y_v + \Phi_{xz} Z_v,$$

and similar expressions hold for Φ_{yv} and Φ_{zv} . Let $2\Omega^*(\xi, \xi')$ be a quadratic form in the variables (ξ, ξ') whose coefficients are the second derivatives of the function $G(x, x')$ with respect to the variables x, y, z, x', y', z' . If we use $v_a = -\zeta/\sigma$, then equation (7:10) may be put in the form indicated in the following theorem:

THEOREM 7:1. The second derivative of the integral sum $I(a)$, evaluated at $a = 0$ for a minimizing surface S_0 whose projection on the xy -plane is the region A_0 , has the value

$$\begin{aligned}
 I''(0) = & \iint_{A_0} 2\omega dx dy + \int_0^{\lambda} \frac{\zeta^2}{\sigma^2} [D_v A + D(A_v + \sigma f_z) + \sum \lambda^* x_{vv}] du \\
 (7:13) & + \int_0^{\lambda} 2\Omega^*(\xi, \xi') du,
 \end{aligned}$$

where $2\Omega^*(\xi, \xi')$ is the quadratic form described above.

When the function f is identically zero, it is seen by equation (6:14) that there exists a function $\uparrow(u)$ such that λ^* , μ^* , ν^* all vanish. The expression (7:13) for the second variation then reduces to

$$I''(0) = \int_0^{\lambda} 2\Omega^*(\xi, \xi') du.$$

If the function g is identically zero, then $B \equiv 0$ and the transversality condition stated in Theorem 6:2 reduces to $A = 0$, where A is defined in the first of equations (7:6). If we make use of the normal coordinate system described above, then by equations (7:12) the expression (7:13) for the second variation becomes

$$I''(0) = \iint_{A_0} 2\omega dx dy + \int_0^{\lambda} \frac{\zeta^2}{\sigma^2} D [P(A_x + Pf_z) + Q(A_y + Qf_z)] du,$$

where $A_x + Pf_z$ and $A_y + Qf_z$ have the values given in equations (7:11). If we set $D = 1$, this result for $I''(0)$ agrees with the equations (49) and (50) defining $I''(0)$ as given in a paper by Simmons [12, p. 248], after a minor correction of equation (50) is made. The sum $P^2(f_p)_x + Q^2(f_q)_y$ which occurs twice in the expression A_2 of equation (50) of that paper should occur only once. It should be noted in correlating the work of this paper with that of Simmons that he made use of a coordinate transformation for which $D = -1$ when $v = 0$.

8. A boundary value problem. A Jacobi condition phrased in terms of a boundary value problem associated with the second variation may be formulated in a manner analogous to that used by Simmons [12, p. 249]. We first express $I''(0)$ in a form involving the Jacobi expression J defined in equation (7:3). To do this, we substitute the value of 2ω given by equation (7:4) into the equation (7:13) with $\bar{T}(u) = 0$ and apply Green's Theorem. By using the equations (6:7) with $a = 0$ we obtain the expression

$$(8:1) \quad I''(0) = \iint_{A_0} \zeta J \, dx \, dy - \int_0^{\lambda} \frac{\zeta}{\sigma} [D(P\omega_\pi + Q\omega_\kappa) - \zeta H(u)] \, du \\ + \int_0^{\lambda} 2\Omega(\xi, \xi') \, du,$$

where

$$H(u) = \frac{1}{\sigma} [D_V A + D(A_V + \sigma f_x) + \sum \lambda X_{VV}].$$

We proceed to express the last written integral in another form.

By means of the homogeneity property

$$2\Omega(\xi, \xi') = \sum \xi_1 \Omega_{\xi_1} + \sum \xi_1' \Omega_{\xi_1'},$$

where the sums are taken for $i = 1, 2, 3$, we obtain

$$(8:2) \quad 2\Omega(\xi, \xi') = \sum \xi_1 \left(\Omega_{\xi_1} - \frac{d}{du} \Omega_{\xi_1'} \right) + \frac{d}{du} \left(\sum \xi_1 \Omega_{\xi_1'} \right).$$

The last expression in the right member of (8:2) vanishes when integrated between the limits 0 and λ , since each term has the period λ in the variable u . By using the relation $v_a = -\zeta/\sigma$, the variations ξ_1, ξ_2, ξ_3 have the values $\xi_1 = -\zeta X_V/\sigma$, $\xi_2 = -\zeta Y_V/\sigma$, $\xi_3 = -\zeta Z_V/\sigma$. With the aid of these relations and the values of $\omega_\pi, \omega_\kappa$ as computed from equation (7:2), the expression (8:1) may be written as

$$\begin{aligned}
 I''(0) = & \iint_{A_0} \zeta J \, dx \, dy - \int_0^{\lambda \zeta} \frac{1}{\sigma} [L \zeta + M \pi + N \kappa \\
 (8:3) \quad & + \sum (\Omega_{\xi_1} - \frac{d}{du} \Omega_{\xi_1},) X_v] du,
 \end{aligned}$$

Where the functions L, M, N are defined by the equations

$$\begin{aligned}
 (8:4) \quad L &= D(Pf_{pz} + Qf_{qz}) - H(u) \\
 M &= D(Pf_{pp} + Qf_{qp}) \\
 N &= D(Pf_{pq} + Qf_{qq}),
 \end{aligned}$$

and $\pi = \zeta_x, \kappa = \zeta_y, P = p - p_1, Q = q - q_1$.

The boundary value problem may now be stated in the following form:

THEOREM 8:1. In order that the surface $z = z(x, y)$ shall minimize the functional I defined in equation (2:3), it is necessary that for negative values of λ^* the boundary value problem

$$\begin{aligned}
 (8:5) \quad J(\zeta) - \lambda^* \zeta &= 0 && \text{in } A_0, \\
 L \zeta + M \pi + N \kappa + \sum (\Omega_{\xi_1} - \frac{d}{du} \Omega_{\xi_1},) X_v &= 0 && \text{on } C_0,
 \end{aligned}$$

have no solution except $\zeta \equiv 0$, where C_0 is the boundary of A_0 . The functions L, M, N are defined in equations (8:4).

The proof that the boundary value problem has no solution except $\zeta \equiv 0$ is made by observing that the line integral in equation (8:3) is zero for every admissible function $\zeta(x, y)$ satisfying the second equation in (8:5), and that a function ζ which satisfies both equations gives $I''(0)$ the value

$$I''(0) = \lambda^* \iint_{A_0} \zeta^2 \, dx \, dy.$$

If ζ were not identically zero, then for $\lambda^* < 0$ the second variation would be negative and the integral sum I would not be minimized.

9. The necessary conditions of Weierstrass and Legendre.

For the problem of minimizing a double integral in a class of surfaces having a common simply closed boundary curve, the necessary conditions of Weierstrass and Legendre are well known [3, pp. 9, 27]. These conditions must also hold along a minimizing surface for the problem of this paper. In addition, it will be shown that along the boundary curve of the minimizing surface a Weierstrass condition formed for the function g must be satisfied. For a problem closely related to that of this paper, Hilbert [9] has pointed out the fact that two Weierstrass conditions play a role; one for the interior of a minimizing surface and one for its boundary.

We shall employ a slight modification of the method used by Graves [8] for establishing the Weierstrass condition for multiple integral variation problems. The procedure consists in setting up two one parameter families of admissible surfaces defined over a small portion of the minimizing surface and a segment of its boundary. When the parameter, which occurs in each of the families, is allowed to approach zero, the entire configuration shrinks to a point on the boundary curve in a manner to be described below.

We consider a rectangular coordinate system in the uv -plane. A square may be defined in this plane bounded by the lines $u = 0$, $u = b$, $v = 0$, $v = b$, where b is a parameter ≥ 0 . Lines passing through the points 0 and 1 respectively, as shown in Figure 1,

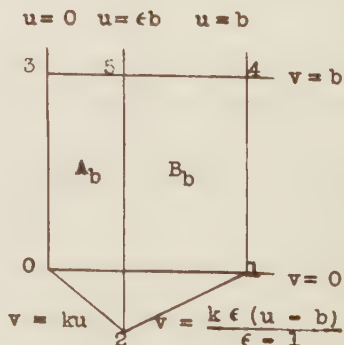


FIGURE 1

are defined by the equations $v = ku$, $v = k\epsilon(u - b)/(\epsilon - 1)$, where $0 < \epsilon < 1$ and k is different from zero. These lines intersect on the line $u = \epsilon b$ at a point 2. In Figure 1, we have for convenience regarded k as negative. Let A_b and B_b denote respectively the areas of the trapezoidal regions 0352 and 1452. Then the values

$$(9:1) \quad A_b = b^2 \epsilon (1 - k\epsilon/2), \quad B_b = b^2(1 - \epsilon)(1 - k\epsilon/2)$$

are easily computed.

The expression for the functional I defined in equation (2:3) may be transformed to uv -coordinates by means of the equations (3:1), which are now assumed to define a normal coordinate system, in the sense described in the last part of Section 7. We consider a portion S_{0b} of the minimizing surface S_0 whose projection on the uv -plane is the region 0143, as shown in Figure 1. Let the remaining part of S_0 be denoted by S_0^* , and let L_0^* be that part of its boundary which is on the fixed surface. For the portion S_{0b} of the minimizing surface we may write

$$I(b) = \int_0^b \int_0^b f D \, dv \, du + \int_0^1 g \, du,$$

where D is defined by equation (3:2) and b is positive and sufficiently near zero, so the coordinate transformation (3:1) is well defined. The arguments of the function g are those belonging to a segment L_{0b} of the boundary curve L_0 of the minimizing surface whose projection on the uv -plane is the line 01. The curve L_{0b} lies on the fixed surface (2:2) and is determined by values $0 \leq u \leq b$, $v = 0$. The equations $v = ku$, $v = k\epsilon(u - b)/(\epsilon - 1)$ also define curves on the fixed surface. These will be denoted by L_{1b} and L_{2b} respectively. It will be shown later in this section that

for each fixed value of the parameter b near zero, an admissible comparison surface S may be constructed which is composed of S_0^* and two surfaces S_{1b} and S_{2b} . The surfaces S_{1b} and S_{2b} form a cap over the portion S_{0b} of the minimizing surface and have as their projections on the uv -plane the regions A_b and B_b respectively. The boundary curve of the comparison surface S is a continuous curve lying on the fixed surface, and consists of the curves L_0^* , L_{1b} , L_{2b} . Define

$$J(b) = \iint_{A_b} fD \, dv \, du + \iint_{B_b} fD \, dv \, du + \int_0^{\epsilon b} g \, du + \int_{\epsilon b}^b g \, du.$$

Here the arguments of the integrand function f in the first double integral are those belonging to the surface S_{1b} and in the second double integral are those belonging to the surface S_{2b} . The first single integral is to be evaluated along the curve L_{1b} and the second along the curve L_{2b} .

Since the surface S_{0b} is part of a minimizing surface for the integral sum defined in equation (2:3), we obtain, after eliminating the value of this sum over the surface S_0^* and the boundary curve L_0^* , the inequality

$$(9:2) \quad J(b) - I(b) \geq 0,$$

which must be satisfied for values of the parameter b sufficiently near zero. This inequality may be written as $K_1(b) + K_2(b) \geq 0$, where

$$K_1(b) = \iint_{A_b} fD \, dv \, du + \iint_{B_b} fD \, dv \, du - \int_0^b \int_0^b fD \, dv \, du,$$

$$K_2(b) = \int_0^{\epsilon b} g \, du + \int_{\epsilon b}^b g \, du - \int_0^b g \, du,$$

and the arguments of the integrand functions in the expressions for $K_1(b)$ and $K_2(b)$ are as described above. If we apply the theorem of mean value for double integrals to $K_1(b)$, then by using equations (9:1) we find an expression of the form $K_1(b) = b^2 H(b)$, where $H(b)$ is a function which is bounded for the parameter b sufficiently small, since the functions f and D are bounded in a neighborhood of the values $u = 0, v = 0$. The function g may be regarded as a function of the arguments u, v, v' . By an application of the theorem of mean value for single integrals to $K_2(b)$, the inequality (9:2) becomes

$$b^2 H(b) + \epsilon b g(u_1, ku_1, k) + b(1 - \epsilon) g\left[u_2, \frac{k\epsilon(u_2 - b)}{\epsilon - 1}, \frac{k\epsilon}{\epsilon - 1}\right] - b g(u_3, 0, 0) \geq 0,$$

where $0 < u_1 < \epsilon b, \epsilon b < u_2 < b, 0 < u_3 < b$. If we divide this inequality by b , let b approach zero, and introduce the original arguments $(X, X_u + X_v v')$ for g , we obtain a form of the Weierstrass condition due to Graves [8]:

$$\epsilon [g(X, X_u + kX_v) - g(X, X_u)] + (1 - \epsilon) [g(X, X_u + \frac{k\epsilon}{\epsilon - 1} X_v) - g(X, X_u)] \geq 0,$$

where the arguments (u, v) are the set $(0, 0)$. One of the usual forms for the Weierstrass condition is obtained if we divide this inequality by ϵ , set $h = \epsilon / (\epsilon - 1)$ and let ϵ approach zero. The E-function has a variety of representations. In our present notation one form is the expression

$$(9:3) \quad E(X, X_u, X') = g(X, X') - g(X, X_u) - \sum (X' - X_u) g_{X'}(X, X_u),$$

where $X' = X_u + kX_v, Y' = Y_u + kY_v, Z' = Z_u + kZ_v, k \neq 0$. As a

consequence of the homogeneity condition imposed on the function g , the E -function may be written in the simpler form

$$E(X, X_u, X') = g(X, X') - \sum X' g_{X'}(X, X_u).$$

At any point P_0 on the boundary curve of a minimizing surface, we consider the lines having the direction parameters (X_u, Y_u, Z_u) and (X_v, Y_v, Z_v) and their projections on the xy -plane. The arguments of the derivatives of X, Y, Z are the set $(u, 0)$. A line with the direction parameters $(X_v, Y_v, 0)$ separates the xy -plane into two parts. Let π be the half-plane in the positive direction of a line with the direction parameters $(X_u, Y_u, 0)$. Then if a line having the direction parameters (X', Y', Z') passes through the point P_0 , lies in the tangent plane to the fixed surface (2:2) at this point, and has as its projection on the xy -plane a line directed interior to the half-plane π , we call (X', Y', Z') an admissible direction. If in place of the positive homogeneity assumption postulated for the function g in Section 2, it is assumed that the stronger homogeneity property $g(x, -x') = g(x, x')$ is satisfied, then every line through the point P_0 and lying in the tangent plane to the fixed surface is said to have an admissible direction. Except for the problem of constructing an admissible comparison surface S , we have justified the following result:

THEOREM 9:1. Let L_0 be the boundary curve of a surface minimizing the functional I defined in equation (2:3). Then the Weierstrass condition

$$E(X, X_u, X') \geq 0$$

must be satisfied at every element (X, X_u) of L_0 for all admissible directions (X', Y', Z') distinct from (X_u, Y_u, Z_u) .

We now consider the problem of constructing the comparison surface S discussed earlier in this section. Two regions, denoted by A_b and B_b , have been defined in the uv -plane. In a neighborhood of $v = 0$ the minimizing surface S_0 has in uv -coordinates the equation

$$z = z[X(u, v), Y(u, v)] = z^*(u, v) \quad (0 \leq u \leq l).$$

We are considering a portion S_{0b} of S_0 defined by the inequalities $0 \leq u \leq b$, $0 \leq v \leq b$. Along the segment L_{0b} of the boundary of this surface the incident relation $z^*(u, 0) = Z(u, 0)$ is satisfied. We regard the parameters k , ϵ , b as fixed, and may define two admissible surfaces S_{1b} and S_{2b} which have as their projections on the uv -plane the regions A_b and B_b respectively. We define the surface S_{1b} by the equation $z = z_1(u, v)$, where for $k < 0$

$$\begin{aligned} z_1(u, v) &= z^*(u, v) \quad \text{when } 0 \leq v \leq b, \\ &= Z(u, v) \quad \text{when } ku \leq v \leq 0, \end{aligned}$$

and where the variable u has the range $0 \leq u \leq \epsilon b$. When $k > 0$, we may define

$$z_1(u, v) = z^*(u, v) + \frac{b-v}{b-ku} [Z(u, ku) - z^*(u, ku)],$$

where $0 \leq u \leq \epsilon b$, $ku \leq v \leq b$, $\epsilon k < 1$. Let $\tau = k\epsilon(u-b)/(\epsilon-1)$.

Then the surface S_{2b} may be defined by the equation $z = z_2(u, v)$, where for $k < 0$

$$\begin{aligned} z_2(u, v) &= z^*(u, v) \quad \text{when } 0 \leq v \leq b, \\ &= Z(u, v) \quad \text{when } \tau \leq v \leq 0, \end{aligned}$$

and the variable u has the range $\epsilon b \leq u \leq b$. When $k > 0$, we may define

$$z_{\epsilon}(u, v) = z^*(u, v) + \frac{b-v}{b-\tau} [Z(u, \tau) - z^*(u, \tau)],$$

where $\epsilon \leq u \leq b$, $\tau \leq v \leq b$, $\epsilon k < 1$. It may be verified that the comparison surface S thus defined has the properties required to carry through the proof of the Weierstrass condition.

We may now establish a Legendre condition as a consequence of the Weierstrass condition stated in Theorem 9:1. The E -function defined in equation (9:3), when considered as a function of k , has a minimum when $k = 0$, and it follows that the relations $E'(0) = 0$, $E''(0) \geq 0$ must be satisfied. The first relation is found to be satisfied identically, and the second relation gives a form of the Legendre condition.

THEOREM 9:2. Let L_0 be the boundary curve of a surface minimizing the functional I defined in equation (2:3). Then at each element (X, X_u) of the boundary curve the Legendre condition

$$(9:4) \quad \begin{aligned} &X_v^2 g_{X'X'} + Y_v^2 g_{Y'Y'} + Z_v^2 g_{Z'Z'} \\ &+ 2(X_v Y_v g_{X'Y'} + Y_v Z_v g_{Y'Z'} + Z_v X_v g_{Z'X'}) \geq 0 \end{aligned}$$

must hold for all directions (X_v, Y_v, Z_v) distinct from (X_u, Y_u, Z_u) and lying in the tangent plane to the fixed surface (2:2) at any point on L_0 . The arguments of the second derivatives of the function g are the set (X, X_u) defining the element on the boundary.

If the fixed surface (2:2) is represented in the non-parametric form $\Phi(x, y, z) = 0$, then Theorem 9:2 may be stated in a slightly different form. It is readily shown that the inequality (9:4) must hold for all directions (X_v, Y_v, Z_v) which are distinct from (X_u, Y_u, Z_u) and which satisfy the orthogonality relation

$$\sum \Phi_x X_v = 0 \text{ along } L_0.$$

If $2\psi(\xi_1; \eta_1)$ represents a bilinear form in the variables ξ_1, ξ_2, ξ_3 and η_1, η_2, η_3 whose coefficients are the second derivatives of g with respect to x', y', z' , then (a) $2\psi(X_u; X_u) = 0$, (b) $2\psi(X_u; X_v) = 0$, and (c) $2\psi(X_v; X_v) \geq 0$. The result (a) is a consequence of the homogeneity condition imposed on the function g . It may be obtained by differentiating the identity $g(X, X_u) = \sum X_u g_{X_i}(X, X_u)$ with respect to X_u, Y_u, Z_u in turn, then multiplying the resulting three equations by X_u, Y_u, Z_u respectively, and adding. A computation similar to that used to derive (a) establishes the relation (b). The relation (c) is merely another notation for the inequality (9:4). This inequality may now be written in the form $2\psi(X'; X') \geq 0$, as is seen from the equation

$$2\psi(X'; X') = 2\psi(X_u; X_u) + 2k \cdot 2\psi(X_u; X_v) + k^2 \cdot 2\psi(X_v; X_v)$$

and the relations (a), (b), (c) above. Hence it follows that if the Legendre condition (9:4) holds for one direction, it holds for all directions of the type indicated.

10. An illustrative example. In this section an application of the results of this paper will be made to the problem of minimizing the expression I defined in equation (2:3) when the functions f and g have the special values

$$(10:1) \quad f = \sqrt{1 + p^2 + q^2}, \quad g = \sqrt{x'^2 + y'^2 + z'^2}.$$

We may assume that the fixed surface is so parametrized that u represents arc length along the boundary curve L_0 of a minimizing surface, and hence $g^2 = x'^2 + y'^2 + z'^2 = 1$ on L_0 . We then find the derivatives $f_p = p/f$, $f_q = q/f$, $\lambda = -x''$, $\mu = -y''$, $\nu = -z''$. The direction cosines of the tangent line to L_0 at any point are $x' = X_u$, $y' = Y_u$, $z' = Z_u$, the arguments being $(u, 0)$. Let (X, m, n)

be the direction cosines of the principal normal to L_0 , and let (a, b, c) represent the direction cosines of the binormal to L_0 . If ρ is the radius of first curvature at any point on L_0 , then we may define the positive direction on the binormal to be such that the relations

$$(10:2) \quad a = \rho(y'z'' - z'y''), \quad b = \rho(z'x'' - x'z''), \quad c = \rho(x'y'' - y'x'')$$

hold at each point of L_0 , where ρ is positive [6]. The positive direction of the principal normal is then fixed if the convention is made that the positive directions of the tangent line, principal normal and binormal at a point have the same mutual orientation as the positive directions of the x -, y -, z -axes, respectively. We then find the relations

$$(10:3) \quad \lambda = \rho x'' = bz' - cy', \quad m = \rho y'' = cx' - az', \quad n = \rho z'' = ay' - bx'.$$

The transversality condition stated in Theorem 6:2 then becomes $D(1 + pp_1 + qq_1)/f + (\sum \lambda x_v)/\rho = 0$. By using equations (10:3) and (6:4), this may be written $(1 + pp_1 + qq_1)/f = (ap_1 + bq_1 - c)/\rho$. Let θ denote the angle between the normal line to the fixed surface and the normal line to the minimizing surface at any point on L_0 , and let ψ denote the angle between the normal line to the fixed surface and the binormal to the curve L_0 . The transversality condition is then

$$(10:4) \quad \rho \cos \theta = \cos \psi \quad (\theta \neq 0).$$

This relation must hold at each point on the boundary curve of a minimizing surface $z = z(x, y)$. A necessary and sufficient condition that the minimizing surface intersect the fixed surface orthogonally is that the boundary curve L_0 be a geodesic on the fixed surface. In this case the normal line to the fixed surface coincides with the principal normal to L_0 , and in addition, the osculating plane

of L_0 coincides with the tangent plane to the minimizing surface at each point of L_0 . Therefore L_0 is an asymptotic curve on the minimizing surface.

The Euler equation for this problem is found by equation (6:9) to be $(1 + q^2)r - 2pq s + (1 + p^2)t = 0$, where p , q and r , s , t are the customary notations for the first and second partial derivatives of $z(x, y)$. The mean curvature of the minimizing surface is therefore zero, and the surface is a minimal surface.

The expression (7:10) for the second variation may be written in the form

$$I''(0) = \iint_{A_0} 2\omega \, dx \, dy + \int_0^{\chi} [2\Omega(\xi, \xi') + \frac{\xi^2}{\sigma^2} C(u)] du,$$

where

$$2\omega(\xi, \eta, \kappa) = [(1 + q^2)\eta^2 - 2pq\eta\kappa + (1 + p^2)\kappa^2]/f^3,$$

$$C(u) = AD_v + D(A_x X_v + A_y Y_v) - \sum X_{uv} X_{vv}.$$

In the expression for $C(u)$ the arguments of the derivatives of X , Y , Z are the set $(u, 0)$. From equation (3:2) and the first equation in (7:6), we find the relations

$$D_v = (X_u Y_{vv} - Y_u X_{vv}) + (Y_v X_{uv} - X_v Y_{uv}),$$

$$A = -(pp_1 + qq_1 + 1)/f.$$

The quadratic form $2\Omega(\xi, \xi')$ reduces to a quadratic form in the three variables ξ_1' , ξ_2' , ξ_3' whose coefficients are the second derivatives of $g = \sqrt{x'^2 + y'^2 + z'^2}$ with respect to x' , y' , z' , and has the explicit expression

$$2\Omega = (\xi_1'^2 + \xi_2'^2 + \xi_3'^2) - (\xi_1' X_u + \xi_2' Y_u + \xi_3' Z_u)^2.$$

By Lagrange's identity, the form 2Ω may be expressed as a sum of squares, and is positive.

The values of A_x and A_y may be found from equations (7:11). We first find the expressions

$$\begin{aligned}(f)_x &= (pr + qs)/f, & (f)_y &= (ps + qt)/f, \\ (f_p)_x &= [(1 + q^2)r - pqs]/f^3, & (f_p)_y &= [(1 + q^2)s - pqt]/f^3, \\ (f_q)_x &= [-pqr + (1 + p^2)s]/f^3, & (f_q)_y &= [-pqs + (1 + p^2)t]/f^3.\end{aligned}$$

We then obtain the values

$$\begin{aligned}A_x &= -(pr_1 + qs_1)/f + r[P(1 + q^2) - Qpq]/f^3 + s[Q(1 + p^2) - Ppq]/f^3, \\ A_y &= -(ps_1 + qt_1)/f + s[P(1 + q^2) - Qpq]/f^3 + t[Q(1 + p^2) - Ppq]/f^3,\end{aligned}$$

where $P = p - p_1$, $Q = q - q_1$. By means of the relations

$$\begin{aligned}P(1 + q^2) - Qpq &= Pf^2 - p(pP + qQ) = -p_1f^2 + p(pp_1 + qq_1 + 1), \\ Q(1 + p^2) - Ppq &= Qf^2 - q(pP + qQ) = -q_1f^2 + q(pp_1 + qq_1 + 1),\end{aligned}$$

the expressions for A_x and A_y may be written in the form

$$\begin{aligned}A_x &= -(pr_1 + qs_1 + rp_1 + sq_1)/f + (pp_1 + qq_1 + 1)(pr + qs)/f^3, \\ A_y &= -(ps_1 + qt_1 + sp_1 + tq_1)/f + (pp_1 + qq_1 + 1)(ps + qt)/f^3.\end{aligned}$$

The functions L , N , N occurring in the boundary value problem stated in Theorem 8:1 have the special values

$$\begin{aligned}L &= -H(u) = -C(u)/\sigma, \\ M &= D[P + q(pq_1 - qp_1)]/f^3, \\ N &= D[Q - p(pq_1 - qp_1)]/f^3.\end{aligned}$$

The Jacobi expression (7:3) has the form

$$J(\zeta) = \frac{\partial}{\partial x} \left[\frac{pq\kappa - (1 + q^2)\pi}{f^3} \right] + \frac{\partial}{\partial y} \left[\frac{pq\pi - (1 + p^2)\kappa}{f^3} \right],$$

For the classical problem of Plateau, $f = \sqrt{1 + p^2 + q^2}$, $g \equiv 0$ and therefore $B \equiv 0$. The transversality condition $A = 0$ is the orthogonality condition $pp_1 + qq_1 + 1 = 0$. The coefficients L, N, N in equation (8:4) have the values

$$\begin{aligned} L &= D[(pr_1 + qs_1 + rp_1 + sq_1)X_v \\ &\quad + (ps_1 + qt_1 + sp_1 + tq_1)Y_v] / \sigma f, \\ M &= -Dp_1/f, \quad N = -Dq_1/f, \end{aligned}$$

where $\sigma = PX_v + QY_v$. If we make use of a normal coordinate system for which $\sigma X_v = P$, $\sigma Y_v = Q$, $\sigma^2 = P^2 + Q^2$, $D = 1$, a simpler form of the Jacobi condition for the problem of Plateau is obtained than that given by Simmons [12, p. 250].

At each element (x, y, z, p, q) of a minimizing surface S_0 the Weierstrass E-function [3, p. 27] has the value

$$E_1(x, y, z, p, q, \bar{p}, \bar{q}) = \frac{\sqrt{1 + p^2 + q^2} \sqrt{1 + \bar{p}^2 + \bar{q}^2} - (p\bar{p} + q\bar{q} + 1)}{\sqrt{1 + p^2 + q^2}},$$

where $(\bar{p}, \bar{q}, -1)$ represents an arbitrary direction at a point on S_0 . The function E_1 is the same as that for the classical problem of Plateau, and is non-negative, vanishing only when $p = \bar{p}$, $q = \bar{q}$. This may readily be shown by considering the numerator of E_1 and applying Lagrange's identity. Along the boundary L_0 of a minimizing surface S_0 the Weierstrass E-function defined in equation (9:3) is found to have the value

$$E_2(X, X_u, X') = \frac{\sqrt{\sum X_u^2} \cdot \sqrt{\sum X'^2} - \sum X_u X'}{\sqrt{\sum X_u^2}},$$

where $X' = X_u + kX_v$, $Y' = Y_u + kY_v$, $Z' = Z_u + kZ_v$. By use of Lagrange's identity the numerator of this expression is readily shown to be non-negative, and to vanish only when $k = 0$. The

direction (X', Y', Z') then coincides with the direction of the tangent line to L_0 .

At every element (x, y, z, p, q) of a minimizing surface S_0 for the integral sum I the Legendre expression

$$f_{pp} \cos^2 \alpha + 2f_{pq} \cos \alpha \sin \alpha + f_{qq} \sin^2 \alpha$$

must be non-negative [3, p. 9] for every value of α . For the special problem of this section this expression may be put in the form $[1 + (p \sin \alpha - q \cos \alpha)^2]/f^3$. The strengthened form of the Legendre condition is therefore satisfied along S_0 .

The Legendre expression given in equation (9:4) is found to have the value $\sum X_v^2 - (\sum X_u X_v)^2$. Let E, F, G denote the fundamental coefficients of the first order for the capstan surface (2:2), and let $H = \sqrt{EG - F^2}$. Then $E = 1$ along L_0 and the Legendre condition states that $H^2 \geq 0$ along L_0 . The inequality $H^2 > 0$ is always satisfied for a real surface without singular points.

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VITA

Edward Alfred Nordhaus was born in Chicago, Illinois, on February 23, 1912. He attended the elementary schools of Chicago, Illinois, and River Forest, Illinois. In June, 1930, he graduated from the Oak Park and River Forest Township High School. As an undergraduate he attended the University of Chicago from 1930-1934, and in June, 1934, received the degree of Bachelor of Science with honors in mathematics. The following year was spent in graduate work in mathematics at the University of Chicago, and in August, 1935, he received the degree of Master of Science. During the academic years 1935-36, 1936-37, 1937-38, he held the position of instructor in mathematics at the University of Wisconsin Extension Division in Milwaukee, and continued his graduate work at the University of Chicago during the summer quarters. He was the recipient of a fellowship in the department of mathematics at the University of Chicago during the academic year 1938-39, and in that period taught four elementary courses. He has attended graduate courses given by Professors Albert, Bartky, Bliss, Dickson, Graves, Lane, Lunn, and Reid, and by visiting lecturers Dribin, Jacobson, and Wilcox. The author wishes to express his indebtedness to all these men, and particularly to Professor Bliss, who suggested the problem discussed in this paper, and to Professor Hestenes, for whose assistance and interest during the preparation of this dissertation he is extremely grateful.

